The Statistical Theory of Turbulent Vorticity

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AMS Tucson AZ, Oct. 26, 2012

Outline

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The Deterministic Navier-Stokes Equations

- A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p \\ u(x,0) &= u_0(x) \end{aligned}$$

with the incompressibility condition

$$\nabla \cdot u = 0,$$

- Using the Reynolds decomposition $U + u$ we get the equation for the large scales in the flow

$$\begin{aligned} U_t + U \cdot \nabla U &= \nu \Delta U + \nabla p + U \cdot \nabla U - \overline{(U+u) \cdot \nabla (U+u)} \\ U(x,0) &= U_0(x) \qquad \text{eddy viscosity} \end{aligned}$$

- The turbulence is quantified by the dimensionless Reynolds number $R = \frac{UL}{\nu}$

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Stochastic Navier-Stokes with Turbulent Noise

- The small scales satisfy a stochastic Naiver-Stokes equation

$$\begin{aligned} du &= (\nu\Delta u - u\cdot\nabla u + \nabla p)dt \\ &+ \sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0}^{M} d_k |k|^{1/3} dt\, e_k(x) \\ &+ u\Big(\sum_{k\neq 0}^{M} \int_{\mathbb{R}} h_k \bar{N}^k(dt,dz)\Big) \\ u(x,0) &= u_0(x) \end{aligned}$$

- Each Fourier component e_k comes with its own Brownian motion b_t^k and deterministic bound $|k|^{1/3}dt$

The Stochastic Vorticity Equation

Taking the curl of the stochastic Navier-Stokes equation and using the vector identity

$$\nabla \times (u \cdot \nabla u) = u \cdot \nabla \omega - \omega \cdot \nabla u + (\nabla \cdot u)\omega = u \cdot \nabla \omega - \omega \cdot \nabla u,$$

and incompressibility, we get the vorticity equation

$$\begin{aligned} \omega_t \quad + & \quad u \cdot \nabla \omega = \nu \Delta \omega + \omega \cdot \nabla u + 2\pi i \sum_{k \neq 0} k \times c_k^{\frac{1}{2}} db_t^k e_k(x) \\ + & \quad 2\pi i \sum_{k \neq 0} k \times d_k |k|^{1/3} dt e_k(x) + \omega \sum_{k \neq 0}^{m} \int_{\mathbb{R}} h_k \bar{N}^k(dt, dz), \\ & \quad \omega(x, 0) = \omega_0(x) \end{aligned}$$

Solution of the Stochastic Vorticity Equation

- We solve (1) using the Feynmann-Kac formula, and Cameron-Martin (or Girsanov's Theorem)
- The solution is

$$
\begin{aligned}
\omega &= e^{Kt} e^{-\int_0^t \nabla u \, dr} e^{\int_0^t dq} M_t \omega^0 + \\
&\quad \sum_{k \neq 0} \int_0^t e^{K(t-s)} e^{-\int_0^t \nabla u \, dr} e^{\int_s^t dq} M_{t-s} \\
&\times \quad (k \times c_k^{1/2} d\beta_s^k + k \times d_k \mu_k ds) e_k(x)
\end{aligned}
$$

- K is the heat operator

$$K = \nu \Delta$$

Cameron-Martin and Feynmann-Kac

- M_t is the Martingale

$$M_t = exp\{-\int_0^t u(B_s, s)\cdot dB_s - \frac{1}{2}\int_0^t |u(B_s, s)|^2 ds\}$$

- Using M_t as an integrating factor eliminates the inertial terms from the equation (1)
- The Feynmann-Kac formula gives the exponential of a sum of terms of the form (log-Poissonian)

$$e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)} = e^{N_t^k \ln\beta + \gamma \ln|k|} = |k|^\gamma \beta^{N_t^k}$$

by a computation similar to the one that produces the geometric Lévy process, see She and Leveque [8]

Independence of Velocity

- The velocity at (x,t) is independent of the vorticity at the same point
- The velocity only depends on the whole vorticity field through the Biot-Savart law

$$u(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\omega(y,t)}{|x-y|^3}\,dy, \tag{1}$$

- We have used the periodicity condition to extend the vorticity field to the whole of $\mathbb{R}^3$
- The independence of $u(x,t)$ of $\omega(x,t)$ is seen by setting $\omega(x,t)=0$, since $\{x\}$ is a set of measure zero the integral in (1) is unchanged

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The Invariant Measure and the Probability Density Functions (PDF)

- The statistical theory of the vorticity dynamics is completely determined by the *invariant measure*, that lives on the infinite-dimensional function space were the vorticity vector resides
- Hopf [4] write down a functional differential equation for the characteristic function of the invariant measure
- The quantity that can be compared directly to experiments is the PDF

$$E(\delta_j u) = E([u(x+s,\cdot) - u(x,\cdot)] \cdot r) = \int_{\infty}^{\infty} x f_j(x) dx,$$

$j = 1$, if $r = \hat{s}$ is the longitudinal direction, and $j = 2$, $r = \hat{t}$, $t \perp s$ is a transversal direction

The Kolmogorov-Hopf Equation

The stochastic vorticity equation is an infinite-dimensional Ito process

$$d(P_t\omega) = (KP_t\omega + D\sum_{k\neq 0}|k|^{1/3}P_t e_k)dt + C^{1/2}\sum_{k\in\mathbb{Z}^3}P_t db_t^k e_k \tag{2}$$

$$P_t = e^{-\int_0^t \nabla u\, dr}\prod_k |k|^{2/3}(2/3)^{N_t^k}M_t$$

The Kolmogorov-Hopf equation for the Ito processes (2) is

$$\frac{\partial\phi}{\partial t} = \frac{1}{2}\mathrm{tr}[P_t C P_t^* \Delta\phi] + \mathrm{tr}[P_t\bar{D}\nabla\phi] + < K(\omega)P_t, \nabla\phi > \tag{3}$$

where $\bar{D} = (|k|^{1/3}D_k)$ and $\phi(\omega)$ is a bounded function of ω

Variance and drift

$$Q_t = \int_0^t e^{K(s)} P_s C P_s^* e^{K^*(s)} ds, \quad E_t = \int_0^t e^{K(s)} P_s \bar{D} ds \tag{4}$$

The solution of the Kolmogorov-Hopf equation (3) is

$$R_t \phi(\omega) = \int_H \phi(e^{Kt} P_t \omega + EI + y) \mathcal{N}_{(0,Q_t)} * \mathbb{P}_{P_t}(dy)$$

Theorem

The invariant measure of the stochastic vorticity equation on $H_c = L^2(\mathbb{T}^3)$ *is,* $\mu(dx) =$

$$e^{<Q^{-1/2}EI,\ Q^{-1/2}x> - \frac{1}{2}|Q^{-1/2}EI|^2} \mathcal{N}_{(0,Q)}(dx) \sum_k \delta_{k,l} \sum_{j=0}^{\infty} p_{m_l}^j \delta_{(N_l - j)}$$

where $Q = Q_\infty$, $E = E_\infty$.

The trouble with Vorticity

- Vorticity may not be continuous although the velocity is
- This is the reason why we use the Hilbert space $L^2(\mathbb{T}^3)$
- We expect the vorticity to lack $2/3$ of a derivative
- One may have to normalize the moments in order to get a finite answer
- Nevertheless with proper normalization we can still project onto well defined PDFs
- The effect of the curl vanishes in the normalization $\lim_{k\to\infty}(Q^{-1/2}E)_k = \lim |k \times d_k||k|^{1/3}/|k \times c_k||k|^{1/3} \to \bar{c}$
- Therefore we still get the same stationary equation (6) for the PDF as for the velocity
- Consequently, the four parameter NIG are also the PDFs for the turbulent vorticity and its moments

The differential equation for the PDF

We can rewrite the Kolmogorov-Hopf equation on the form

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{tr}[Q_t \Delta \phi] + \mathrm{tr}[E_t \nabla \phi] \tag{5}$$

Then by scaling $Q^{-1/2}E$ and taking the trace, we get

$$\frac{1}{2}\phi_{rr} + \frac{1+|c|}{r}\phi_r = \frac{1}{2}\phi \tag{6}$$

This is the stationary equation satisfied by the PDF

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The Probability Density Function (PDF)

Lemma

The PDF is a Normalized Inverse Gaussian distribution NIG of Barndorff-Nilsen [1]:

$$f(x_j) = \frac{(\delta/\gamma)}{\sqrt{2\pi} K_1(\delta\gamma)} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x_j - \mu)^2}\right) e^{\beta(x-\mu)}}{\left(\sqrt{\delta^2 + (x_j - \mu)^2}/\alpha\right)} \tag{7}$$

where K_1 is modified Bessel's function of the second kind, $\gamma = \sqrt{\alpha^2 - \beta^2}$.

$$f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{\Gamma(1) 2 e^{\beta\mu}}{(\delta^2 + (x-\mu)^2)}, \quad \text{for } x << 1$$

$$f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{e^{\beta(x-\mu)} e^{-\alpha x}}{x^{3/2}}, \quad \text{for } x >> 1$$

Inserting a Gaussian

- The probability density function (PDF) of the components of the velocity increments is a nomalized inverse Gaussian distribution, see Barndorff-Nilsen [1]
- Letting $\alpha, \delta \to \infty$, in the formulas for $f_j(x)$ below, in such a way that $\delta/\alpha \to \sigma$, we get that

$$f_j \to \frac{e^{-\frac{(x-\mu)^2}{2\sigma}}}{\sqrt{2\pi\sigma}} e^{\beta(x-\mu)}.$$

- The exponential tails of the PDF are caused by occasional sharp velocity gradients (rounded of shocks)
- The cusp at the origin is caused by the random and gentile fluid motion in the center of the ramps leading up to the sharp velocity gradients, see Kraichnan [6]

Outline

Existence and Uniqueness of the Invariant Measure

- We now compare the above PDFs with the PDFs found in simulations and experiments.
- The direct Navier-Stokes (DNS) simulations were provided by Michael Wilczek from his Ph.D. thesis, see [9].
- The experimental results are from Eberhard Bodenschatz experimental group in Göttingen.
- We thank both for the permission to use these results to compare with the theoretically computed PDFs.
- A special case of the hyperbolic distribution, the NIG distribution, was used by Barndorff-Nilsen, Blaesild and Schmiegel [2] to obtain fits to the PDFs for three different experimental data sets.

The PDF from simulations and fits for the longitudinal direction

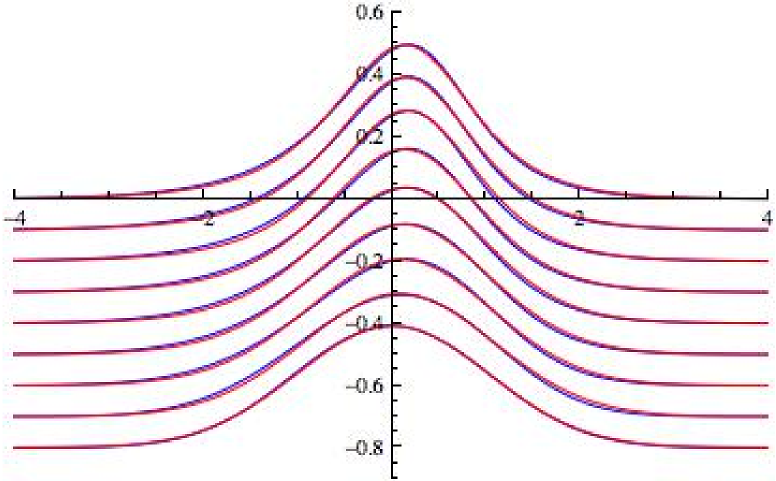

Figure: The PDF from simulations and fits for the longitudinal direction.

The log of the PDF from simulations and fits for the longitudinal direction

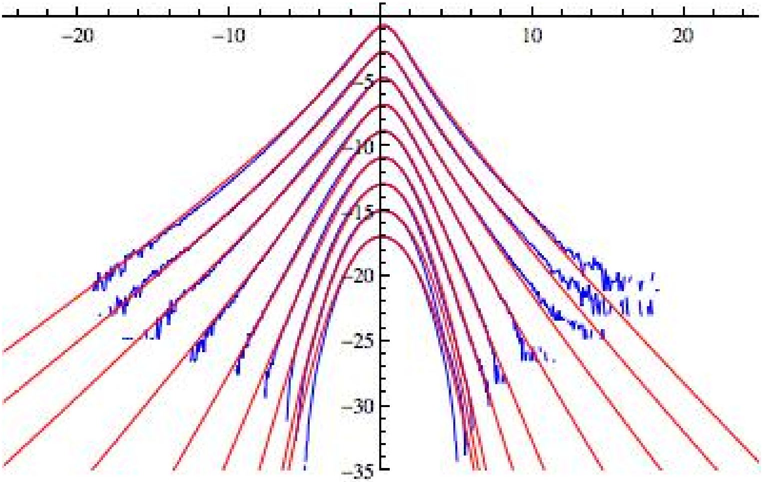

Figure: The log of the PDF from simulations and fits for the longitudinal direction, compare Fig. 4.5 in [9].

The PDF from simulations and fits for the transversal direction

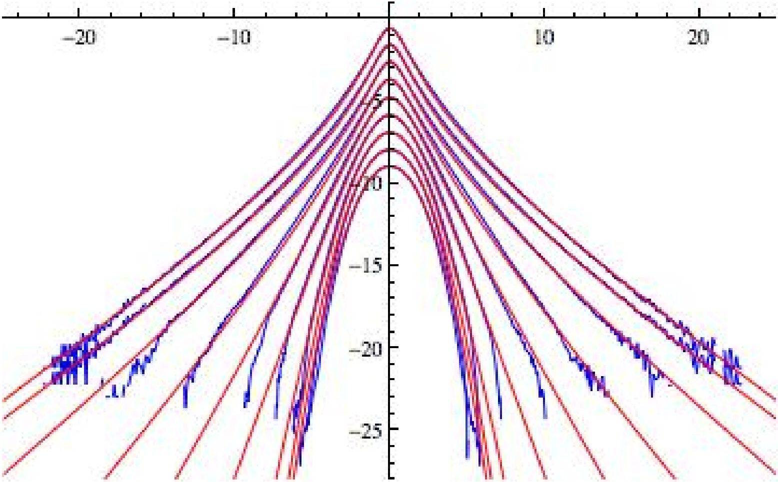

Figure: The log of the PDF from simulations and fits for the a

The PDFs from experiments and fits

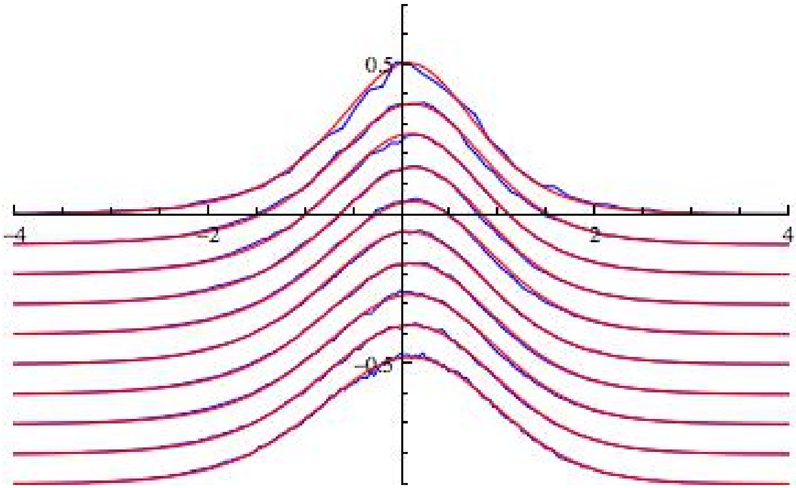

Figure: The PDFs from experiments and fits.

The log of the PDFs from experiments and fits

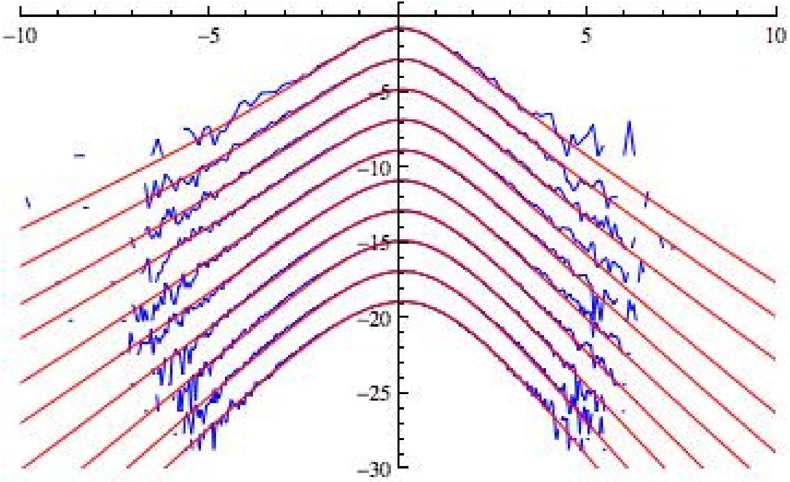

Figure: The log of the PDFs from experiments and fits.

Conclusions

- Given the stochastic Navier-Stokes we can find an equation for the stochastic vorticity
- This equation is linear in ω and can be solved explicitly in terms of u
- This allows us to view vorticity as an infinite dimensional Ito-Lévy process
- We can find the Kolmogorov-Hopf equation for this process and solve for the invariant measure
- The invariant measure can be projected to the PDF that is a Normalized Inverse Gaussian (NIG)
- The comparison with simulated and experimental PDF is excellent

The Artist by the Water's Edge
Leonardo da Vinci Observing Turbulence

Computation of the structure functions

Lemma (The Kolmogorov-Obukov scaling)

The scaling of the structure functions is

$$S_p \sim C_p|x-y|^{\zeta_p},$$

where

$$\zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3})$$

$\frac{p}{3}$ *being the Kolmogorov scaling and* τ_p *the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's* $4/5$ *law,*

$$S_3 = -\frac{4}{5}\varepsilon|x-y|$$

to leading order, were $\varepsilon = \frac{d\mathcal{E}}{dt}$ *is the energy dissipation*

The first few structure functions

$$S_1(x,y,t) = \frac{2}{C}\sum_{k\in\mathbb{Z}^3\backslash\{0\}} d_k \frac{(1-e^{-\lambda_k t})}{|k|^{\zeta_1}} \sin(\pi k\cdot(x-y)).$$

$\sum_{k\in\mathbb{Z}^3\backslash\{0\}} d_k < \infty$, and for $|x-y|$ small,

$$S_1(x,y,\infty) \sim \frac{2}{C}\sum_{k\in\mathbb{Z}^3\backslash\{0\}} d_k |x-y|^{\zeta_1},$$

where $\zeta_1 = 1/3+\tau_1 \approx 0.37$. Similarly

$$S_2(x,y,\infty) \sim \frac{2\pi^{\zeta_2}}{C}\sum_{k\in\mathbb{Z}^3} [c_k + \frac{2d_k^2}{C}]|x-y|^{\zeta_2},$$

when $|x-y|$ is small, where $\zeta_2 = 2/3+\tau_2 \approx 0.696$.

The higher order structure functions

All the structure functions are computed in a similar manner. If $p = 2n+1$ is odd,

$$S_p = \frac{2^p}{C^p} \sum_{k \in \mathbb{Z}^3} {d_k}^p \frac{(1-e^{-2\lambda_k t})^p}{|k|^{\zeta_p}} \sin^n(\pi k \cdot (x-y))$$

to leading order in the lag variable $|x-y|$. If $p = 2n$ is even, S_p is

$$\sum_{k \in \mathbb{Z}^3} [\frac{2^n}{C^n} c_k^n \frac{(1-e^{-2\lambda_k t})^n}{|k|^{\zeta_p}} + \frac{2^p}{C^p} {d_k}^p \frac{(1-e^{-\lambda_k t})^p}{|k|^{\zeta_p}}] \sin^p(\pi k \cdot (x-y)),$$

to leading order in $|x-y|$.

The Kolmogorov-Obukov scaling hypothesis

- The Kolmogorov-Obukov scaling *with the intermittency corrections* τ_p, is

$$S_n(l) = C_p l^{\zeta_p}, \quad \zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3}) \quad (8)$$

where l is the lag variable $l = |x-y|$.

- The coefficients C_p are not universal but depend on the c_ks and d_ks that in turn depend on the large eddies in the turbulent flow
- $C_p = \frac{2^p \pi^{\zeta_p}}{C^p} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} d_k{}^p$ or $C_p = \frac{2^n \pi^{\zeta_p}}{C^n} \sum_{k\in\mathbb{Z}^3} [c_k^n + \frac{2^n}{C^n} d_k{}^p]$

Kolmogorov's refined scaling hypothesis

- In [5, 7] Kolmogorov and Obukhov presented their refined similarity hypothesis

$$S_p = C_p' < \tilde{\varepsilon}^p > l^{p/3}$$

where l is the lag variable and $\tilde{\varepsilon}$ is an averaged energy dissipation rate

- It can be shown, see [3], that by defining $\tilde{\varepsilon}$ appropriately, this gives

$$S_p = C_p' < \tilde{\varepsilon}^p > l^{p/3} = C_p l^{\zeta_p}$$

where the coefficients C_p' now *are universal*

- $$S_p(t,T,l) = C_p l^{\zeta_p} + D_p(t) T^{\gamma_p},\ \gamma_p = \frac{p}{6} + 3(1-(2/3)^{p/3})$$

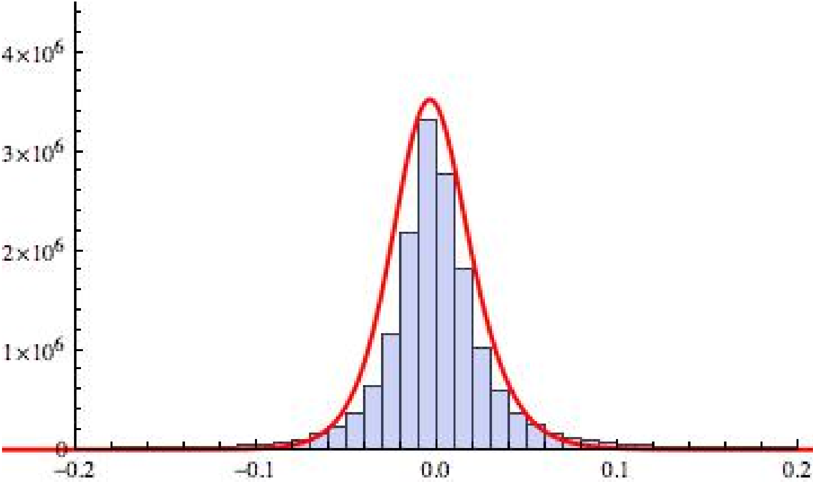

Figure: The PDF for the first structure function, from experiments and fits

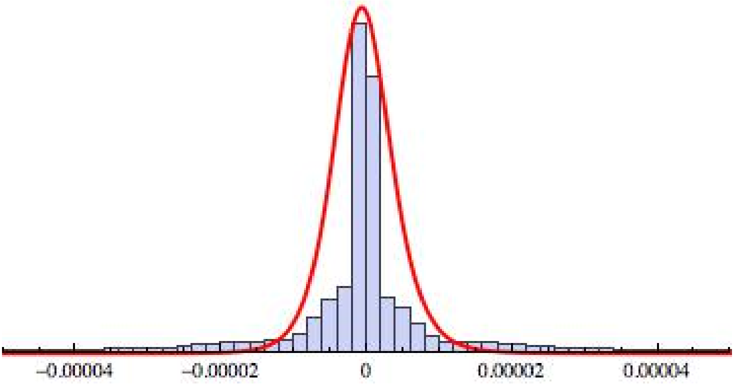

Figure: The PDF for the third structure function, from experiments and fits.

Computing the PDF from the characteristic function

- Taking the characteristic functions of the measure of the stochastic processes, we get

$$\hat{f}(k) \sim k^{1-\zeta_1} e^{-\delta k}$$

- Translating this function and taking the inverse Fourier transform gives

$$f(x) \sim \frac{e^{-d|x|} e^{-bx}}{(x - i\delta)^{2-\zeta_1}}$$

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