The Statistical Theory of Turbulent Vorticity

Björn Birnir

Center for Complex and Nonlinear Science
and
Department of Mathematics, UC Santa Barbara

AMS Tucson AZ, Oct. 26, 2012
Outline

1. The Deterministic versus the Stochastic Equation
2. The Form of the Noise
3. The Kolmogorov-Hopf Equation and the Invariant Measure
4. The Normalized Inverse Gaussian (NIG) distributions
5. Comparison with Simulations and Experiments.
Turbulence
Birnir

Outline

1 The Deterministic versus the Stochastic Equation

2 The Form of the Noise

3 The Kolmogorov-Hopf Equation and the Invariant Measure

4 The Normalized Inverse Gaussian (NIG) distributions

5 Comparison with Simulations and Experiments.
The Deterministic Navier-Stokes Equations

- A general incompressible fluid flow satisfies the Navier-Stokes Equation

\[ u_t + u \cdot \nabla u = \nu \Delta u - \nabla p \]

\[ u(x, 0) = u_0(x) \]

with the incompressibility condition

\[ \nabla \cdot u = 0, \]

- Using the Reynolds decomposition \( U + u \) we get the equation for the large scales in the flow

\[ U_t + U \cdot \nabla U = \nu \Delta U + \nabla p + U \cdot \nabla U - (U + u) \cdot \nabla (U + u) \]

\[ U(x, 0) = U_0(x) \quad \text{eddy viscosity} \]

- The turbulence is quantified by the dimensionless Reynolds number \( R = \frac{UL}{\nu} \)
Outline

1. The Deterministic versus the Stochastic Equation
2. The Form of the Noise
3. The Kolmogorov-Hopf Equation and the Invariant Measure
4. The Normalized Inverse Gaussian (NIG) distributions
5. Comparison with Simulations and Experiments.
The small scales satisfy a stochastic Navier-Stokes equation

\[
\text{du} = (\nu \Delta u - u \cdot \nabla u + \nabla p) dt \\
+ \sum_{k \neq 0} c_k^{1/2} d b_t^k e_k(x) + \sum_{k \neq 0} d_k |k|^{1/3} dt e_k(x) \\
+ u(\sum_{k \neq 0} \int_{\mathbb{R}} h_k \tilde{N}^k(dt, dz))
\]

\[
u \Delta u = -u \cdot \nabla u + \nabla p + \sum_{k \neq 0} c_k^{1/2} d b_t^k e_k(x) + \sum_{k \neq 0} d_k |k|^{1/3} dt e_k(x) + u(\sum_{k \neq 0} \int_{\mathbb{R}} h_k \tilde{N}^k(dt, dz))
\]

\[
u \Delta u = -u \cdot \nabla u + \nabla p + \sum_{k \neq 0} c_k^{1/2} d b_t^k e_k(x) + \sum_{k \neq 0} d_k |k|^{1/3} dt e_k(x) + u(\sum_{k \neq 0} \int_{\mathbb{R}} h_k \tilde{N}^k(dt, dz))
\]

\[u(x, 0) = u_0(x)\]

Each Fourier component \(e_k\) comes with its own Brownian motion \(b_t^k\) and deterministic bound \(|k|^{1/3} dt\)
The Stochastic Vorticity Equation

Taking the curl of the stochastic Navier-Stokes equation and using the vector identity

\[ \nabla \times (u \cdot \nabla u) = u \cdot \nabla \omega - \omega \cdot \nabla u + (\nabla \cdot u)\omega = u \cdot \nabla \omega - \omega \cdot \nabla u, \]

and incompressibility, we get the vorticity equation

\[ \omega_t + u \cdot \nabla \omega = \nu \Delta \omega + \omega \cdot \nabla u + 2\pi i \sum_{k \neq 0} k \times c_k^2 dB_t^k e_k(x) \]

\[ + 2\pi i \sum_{k \neq 0} k \times d_k |k|^{1/3} dte_k(x) + \omega \sum_{k \neq 0} \int_{\mathbb{R}} h_k \tilde{N}^k(dt, dz), \]

\[ \omega(x, 0) = \omega_0(x) \]
Solution of the Stochastic Vorticity Equation

- We solve (1) using the Feynmann-Kac formula, and Cameron-Martin (or Girsanov’s Theorem)

- The solution is

\[
\omega = e^{Kt} e^{-\int_0^t \nabla u \, dr} e^{\int_0^t dq \, M_t} \omega^0 + \sum_{k \neq 0} \int_0^t e^{K(t-s)} e^{-\int_0^s \nabla u \, dr} e^{\int_s^t dq \, M_{t-s}} \times (k \times c_k^{1/2} d\beta^k_s + k \times d_k \mu_k ds) e_k(x)
\]

- \( K \) is the heat operator

\[
K = \nu \Delta
\]
Cameron-Martin and Feynmann-Kac

- $M_t$ is the Martingale

$$M_t = \exp\left\{ -\int_0^t u(B_s, s) \cdot dB_s - \frac{1}{2} \int_0^t |u(B_s, s)|^2 ds \right\}$$

- Using $M_t$ as an integrating factor eliminates the inertial terms from the equation (1)

- The Feynmann-Kac formula gives the exponential of a sum of terms of the form (log-Poissonian)

$$e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k)N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)} = e^{N_t^k \ln \beta + \gamma \ln |k|} = |k| \gamma \beta^{N_t^k}$$

by a computation similar to the one that produces the geometric Lévy process, see She and Leveque [8]
Independence of Velocity

- The velocity at \((x, t)\) is independent of the vorticity at the same point.
- The velocity only depends on the whole vorticity field through the Biot-Savart law

\[
    u(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \times \omega(y, t)}{|x - y|^3} \, dy,
\]

(1)

- We have used the periodicity condition to extend the vorticity field to the whole of \(\mathbb{R}^3\).
- The independence of \(u(x, t)\) of \(\omega(x, t)\) is seen by setting \(\omega(x, t) = 0\), since \(\{x\}\) is a set of measure zero the integral in (1) is unchanged.
Outline

1. The Deterministic versus the Stochastic Equation
2. The Form of the Noise
3. The Kolmogorov-Hopf Equation and the Invariant Measure
4. The Normalized Inverse Gaussian (NIG) distributions
5. Comparison with Simulations and Experiments.
The statistical theory of the vorticity dynamics is completely determined by the *invariant measure*, that lives on the infinite-dimensional function space were the vorticity vector resides.

- The quantity that can be compared directly to experiments is the PDF

\[
E(\delta_j u) = E([u(x + s, \cdot) - u(x, \cdot)] \cdot r) = \int_{-\infty}^{\infty} xf_j(x)dx,
\]

\(j = 1\), if \(r = \hat{s}\) is the longitudinal direction, and \(j = 2\), \(r = \hat{t}\), \(t \perp s\) is a transversal direction.
The stochastic vorticity equation is an infinite-dimensional Ito process

\[ d(P_t \omega) = (KP_t \omega + D \sum_{k \neq 0} |k|^{1/3} P_t e_k) dt + C^{1/2} \sum_{k \in \mathbb{Z}^3} P_t dB_t^k e_k \]

\[ P_t = e^{-\int_0^t \nabla u \, dr} \prod_{k} |k|^{2/3} (2/3)^{N_t^k} M_t \]

The Kolmogorov-Hopf equation for the Ito processes (2) is

\[ \frac{\partial \phi}{\partial t} = \frac{1}{2} \text{tr}[P_t CP_t^* \Delta \phi] + \text{tr}[P_t \bar{D} \nabla \phi] + \langle K(\omega) P_t, \nabla \phi \rangle \]

where \( \bar{D} = (|k|^{1/3} D_k) \) and \( \phi(\omega) \) is a bounded function of \( \omega \)
The Invariant Measure of the Stochastic Vorticity Equation

Variance and drift

\[ Q_t = \int_0^t e^{K(s)} P_s CP_s^* e^{K^*(s)} ds, \quad E_t = \int_0^t e^{K(s)} P_s \bar{D} ds \]  

The solution of the Kolmogorov-Hopf equation (3) is

\[ R_t \phi(\omega) = \int_H \phi(e^{Kt} P_t \omega + EI + y) \mathcal{N}(0, Q_t) * \mathcal{P} P_t(dy) \]

Theorem

The invariant measure of the stochastic vorticity equation on 
\( H_c = L^2(\mathbb{T}^3) \) is, \( \mu(dx) = \)

\[ e^{<Q^{-1/2}EI, Q^{-1/2}x>-\frac{1}{2}|Q^{-1/2}EI|^2} \mathcal{N}(0, Q)(dx) \sum_k \delta_{k,l} \sum_{j=0}^\infty p_{m,l}^j \delta(N_l - j) \]

where \( Q = Q_\infty, E = E_\infty \).
The trouble with Vorticity

- Vorticity may not be continuous although the velocity is.
- This is the reason why we use the Hilbert space $L^2(T^3)$.
- We expect the vorticity to lack $2/3$ of a derivative.
- One may have to normalize the moments in order to get a finite answer.
- Nevertheless with proper normalization we can still project onto well defined PDFs.
- The effect of the curl vanishes in the normalization:
  $$\lim_{k \to \infty} (Q^{-1/2}E)_k = \lim |k \times d_k| |k|^{1/3}/|k \times c_k| |k|^{1/3} \to \bar{c}$$
- Therefore we still get the same stationary equation (6) for the PDF as for the velocity.
- Consequently, the four parameter NIG are also the PDFs for the turbulent vorticity and its moments.
We can rewrite the Kolmogorov-Hopf equation on the form
\[
\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{tr}[Q_t \Delta \phi] + \text{tr}[E_t \nabla \phi]
\] (5)

Then by scaling \( Q^{-1/2} E \) and taking the trace, we get
\[
\frac{1}{2} \phi_{rr} + \frac{1 + |c|}{r} \phi_r = \frac{1}{2} \phi
\] (6)

This is the stationary equation satisfied by the PDF.
Outline

1. The Deterministic versus the Stochastic Equation
2. The Form of the Noise
3. The Kolmogorov-Hopf Equation and the Invariant Measure
4. The Normalized Inverse Gaussian (NIG) distributions
5. Comparison with Simulations and Experiments.
The Probability Density Function (PDF)

Lemma

The PDF is a Normalized Inverse Gaussian distribution NIG of Barndorff-Nilsen [1]:

\[
f(x_j) = \frac{(\delta/\gamma)}{\sqrt{2\pi K_1(\delta\gamma)}} \frac{K_1 \left( \alpha \sqrt{\delta^2 + (x_j - \mu)^2} \right)}{\left( \sqrt{\delta^2 + (x_j - \mu)^2 / \alpha} \right)} e^{\beta (x - \mu)}
\]

where \( K_1 \) is modified Bessel’s function of the second kind, \( \gamma = \sqrt{\alpha^2 - \beta^2} \).

\[
f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{\Gamma(1)2e^{\beta\mu}}{\left( \delta^2 + (x - \mu)^2 \right)}, \quad \text{for } x << 1
\]

\[
f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{e^{\beta (x - \mu)} e^{-\alpha x}}{x^{3/2}}, \quad \text{for } x >> 1
\]
The probability density function (PDF) of the components of the velocity increments is a normalized inverse Gaussian distribution, see Barndorff-Nilsen [1].

Letting $\alpha, \delta \to \infty$, in the formulas for $f_j(x)$ below, in such a way that $\delta/\alpha \to \sigma$, we get that

$$f_j \to e^{-\frac{(x-\mu)^2}{2\sigma}} \frac{e^{\beta(x-\mu)}}{\sqrt{2\pi\sigma}}.$$

The exponential tails of the PDF are caused by occasional sharp velocity gradients (rounded of shocks).

The cusp at the origin is caused by the random and gentile fluid motion in the center of the ramps leading up to the sharp velocity gradients, see Kraichnan [6].
Outline

1. The Deterministic versus the Stochastic Equation
2. The Form of the Noise
3. The Kolmogorov-Hopf Equation and the Invariant Measure
4. The Normalized Inverse Gaussian (NIG) distributions
5. Comparison with Simulations and Experiments.
We now compare the above PDFs with the PDFs found in simulations and experiments.

The direct Navier-Stokes (DNS) simulations were provided by Michael Wilczek from his Ph.D. thesis, see [9].

The experimental results are from Eberhard Bodenschatz experimental group in Göttingen.

We thank both for the permission to use these results to compare with the theoretically computed PDFs.

A special case of the hyperbolic distribution, the NIG distribution, was used by Barndorff-Nilsen, Blaesild and Schmiegel [2] to obtain fits to the PDFs for three different experimental data sets.
The PDF from simulations and fits for the longitudinal direction.

**Figure:** The PDF from simulations and fits for the longitudinal direction.
The log of the PDF from simulations and fits for the longitudinal direction

**Figure:** The log of the PDF from simulations and fits for the longitudinal direction, compare Fig. 4.5 in [9].
The PDF from simulations and fits for the transversal direction

**Figure:** The log of the PDF from simulations and fits for the transversal direction. Compare Fig. 4.6 in [9].
The PDFs from experiments and fits

Figure: The PDFs from experiments and fits.
The log of the PDFs from experiments and fits.
Conclusions

- Given the stochastic Navier-Stokes we can find an equation for the stochastic vorticity.
- This equation is linear in $\omega$ and can be solved explicitly in terms of $u$.
- This allows us to view vorticity as an infinite dimensional Ito-Lévy process.
- We can find the Kolmogorov-Hopf equation for this process and solve for the invariant measure.
- The invariant measure can be projected to the PDF that is a Normalized Inverse Gaussian (NIG).
- The comparison with simulated and experimental PDF is excellent.
The Artist by the Water’s Edge
Leonardo da Vinci Observing Turbulence

Turbulence
Birnir

The Deterministic versus the Stochastic Equation
The Form of the Noise
The Kolmogorov-Hopf Equation and the Invariant Measure
The Normalized Inverse Gaussian (NIG) distributions
Comparison with Simulations and Experiments.
Lemma (The Kolmogorov-Obukov scaling)

The scaling of the structure functions is

\[ S_p \sim C_p |x - y|^{\zeta_p}, \]

where

\[ \zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2 \left(1 - \left(\frac{2}{3}\right)^{p/3}\right) \]

\( \frac{p}{3} \) being the Kolmogorov scaling and \( \tau_p \) the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's 4/5 law,

\[ S_3 = -\frac{4}{5} \varepsilon |x - y| \]

to leading order, were \( \varepsilon = \frac{dE}{dt} \) is the energy dissipation.
The first few structure functions

\[ S_1(x, y, t) = \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k \frac{(1 - e^{-\lambda_k t})}{|k|^{\zeta_1}} \sin(\pi k \cdot (x - y)). \]

\[ \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k < \infty, \text{ and for } |x - y| \text{ small,} \]

\[ S_1(x, y, \infty) \sim \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k |x - y|^{\zeta_1}, \]

where \( \zeta_1 = 1/3 + \tau_1 \approx 0.37. \) Similarly

\[ S_2(x, y, \infty) \sim \frac{2\pi^{\zeta_2}}{C} \sum_{k \in \mathbb{Z}^3} \left[ c_k + \frac{2d_k^2}{C} \right] |x - y|^{\zeta_2}, \]

when \( |x - y| \) is small, where \( \zeta_2 = 2/3 + \tau_2 \approx 0.696. \)
The higher order structure functions

All the structure functions are computed in a similar manner. If $p = 2n + 1$ is odd,

$$S_p = \frac{2^p}{C^p} \sum_{k \in \mathbb{Z}^3} d_k^p \left(1 - e^{-2\lambda_k t}\right)^p \frac{\sin^n(\pi k \cdot (x - y))}{|k|^{\zeta_p}}$$

to leading order in the lag variable $|x - y|$. If $p = 2n$ is even, $S_p$ is

$$\sum_{k \in \mathbb{Z}^3} \left[\frac{2^n}{C^n} c_k^n \frac{(1 - e^{-2\lambda_k t})^n}{|k|^{\zeta_p}} + \frac{2^p}{C^p} d_k^p \frac{(1 - e^{-\lambda_k t})^p}{|k|^{\zeta_p}}\right] \sin^p(\pi k \cdot (x - y)),$$

to leading order in $|x - y|$. 
The Kolmogorov-Obukov scaling hypothesis

- The Kolmogorov-Obukov scaling with the intermittency corrections $\tau_p$, is

$$S_n(l) = C_p l^{\zeta_p}, \quad \zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1 - (2/3)^{p/3}) \quad (8)$$

where $l$ is the lag variable $l = |x - y|$.  

- The coefficients $C_p$ are not universal but depend on the $c_k$s and $d_k$s that in turn depend on the large eddies in the turbulent flow

$$C_p = \frac{2^p \pi^{\zeta_p}}{C_p} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k^p \text{ or } C_p = \frac{2^n \pi^{\zeta_p}}{C^n} \sum_{k \in \mathbb{Z}^3} [c_k^n + \frac{2^n}{C^n} d_k^p]$$
Kolmogorov’s refined scaling hypothesis

- In [5, 7] Kolmogorov and Obukhov presented their refined similarity hypothesis

\[ S_p = C'_p \langle \tilde{\varepsilon}^p \rangle \bar{l}^{p/3} \]

where \( l \) is the lag variable and \( \tilde{\varepsilon} \) is an averaged energy dissipation rate.

- It can be shown, see [3], that by defining \( \tilde{\varepsilon} \) appropriately, this gives

\[ S_p = C'_p \langle \tilde{\varepsilon}^p \rangle \bar{l}^{p/3} = C_p l^{\tilde{\gamma}_p} \]

where the coefficients \( C'_p \) now are universal.

- \( S_p(t, T, l) = C_p l^{\tilde{\gamma}_p} + D_p(t) T^{\gamma_p} \), \( \gamma_p = \frac{p}{6} + 3(1 - (2/3)^{p/3}) \)
Figure: The PDF for the first structure function, from experiments and fits.
Figure: The PDF for the third structure function, from experiments and fits.
Computing the PDF from the characteristic function

- Taking the characteristic functions of the measure of the stochastic processes, we get
  \[ \hat{f}(k) \sim k^{1-\zeta_1} e^{-\delta k} \]

- Translating this function and taking the inverse Fourier transform gives
  \[ f(x) \sim \frac{e^{-d|x|} e^{-bx}}{(x - i\delta)^{2-\zeta_1}} \]


E. Hopf.
Statistical hydrodynamics and functional calculus. 

A. N. Kolmogorov.  
A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. 

R. H. Kraichnan.  
Turbulent cascade and intermittency growth. 

A. M. Obukhov.  
Some specific features of atmospheric turbulence. 