The Kolmogorov-Obukhov Statistical Theory of Turbulence

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The Deterministic versus the Stochastic Equation

The Form of the Noise in Fully Developed Turbulence

The Kolmogorov-Obukhov-She-Leveque Scaling

The Invariant Measure of Turbulence

The Normalized Inverse Gaussian (NIG) distributions

Outline

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The Deterministic Navier-Stokes Equations

- A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p \\ u(x,0) &= u_0(x) \end{aligned}$$

with the incompressibility condition

$$\nabla \cdot u = 0,$$

- Eliminating the pressure using the incompressibility condition gives

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 \\ u(x,0) &= u_0(x) \end{aligned}$$

- The turbulence is quantified by the dimensionless Taylor-Reynolds number $Re_\lambda = \frac{U\lambda}{\nu}$

Reynolds Decomposition

- The stochastic equation below is the equation describing the *small scale flow*
- The velocity is written as $U+u$
- U describes the large scale flow, u describes the small scale turbulence
- This is just the classical Reynolds decomposition

$$\begin{aligned} U_t + U\cdot\nabla U &= \nu\Delta U - \nabla p - (\overline{((U+u)\cdot\nabla(U+u)} - U\cdot\nabla U) \\ &+ f \end{aligned}$$

- The last term before the forcing f, the eddy viscosity, describes how the small scale influence the large ones

Stochastic Navier-Stokes with Turbulent Noise

- Adding the two types of additive noise and the multiplicative noise we get the stochastic Navier-Stokes equations describing fully developed turbulence

$$
\begin{aligned}
du &= (\nu\Delta u - u\cdot\nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt \\
&+ \sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0} d_k |k|^{1/3} dt\ e_k(x) \\
&+ u(\sum_{k\neq 0}^{M} \int_{\mathbb{R}} h_k \bar{N}^k(dt,dz)) \\
u(x,0) &= u_0(x)
\end{aligned}
$$

- Each Fourier component e_k comes with its own Brownian motion b_t^k and deterministic bound $|k|^{1/3}dt$

Outline

The central limit theorem

- Split the torus $\mathbb{T}^3$ into little boxes and consider the dissipation to be a stochastic process in each box
- By the central limit theorem the scaled average

$$M_n = \frac{1}{n}\sum_{j=1}^{n} p_j$$

$\sqrt{n}(M_n - m)/\sigma \to N(0,1)$ converges to a Gaussian (normal) random variable as $n \to \infty$

- This holds for any Fourier component (e_k) and the result is the infinite dimensional Brownian motion

$$df_t^1 = \sum_{k \neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x)$$

Fluctuations and the large deviation principle

- In addition we get fluctuation in the mean of the dissipation
- If these fluctuation are completely random then they are modeled by Poisson process with the rate μ
- Applying the large deviation principle, get a deterministic bound, with rate μ_k
- This also holds in the direction of each Fourier component and gives Fourier series

$$df_t^2 = \sum_{k \neq 0} d_k \mu_k dt \; e_k(x), \;\; \mu_k = |k|^{1/3}$$

Intermittency and velocity fluctuation

- The multiplicative noise, models the excursion (jumps) in the velocity gradient (vorticity concentrations)
- N_t^k denotes the integer number of velocity excursion, associated with kth wavenumber, that have occurred at time t.
- The differential $dN^k(t) = N^k(t+dt) - N^k(t)$ denotes these excursions in the time interval $(t, t+dt]$.
- The process

$$df_t^3 = \sum_{k \neq 0}^{M} \int_{\mathbb{R}} h_k(t,z) \bar{N}^k(dt, dz),$$

gives the multiplicative noise term

Solution of the Stochastic Navier-Stokes

- We solve (1) using the Feynmann-Kac formula, and Cameron-Martin (or Girsanov's Theorem)
- The solution is

$$
\begin{aligned}
u &= e^{Kt} e^{\int_0^t dq} M_t u^0 \\
&+ \sum_{k \neq 0} \int_0^t e^{K(t-s)} e^{\int_s^t dq} M_{t-s} (c_k^{1/2} d\beta_s^k + d_k \mu_k ds) e_k(x)
\end{aligned}
$$

- K is the operator $K = \nu\Delta + \nabla\Delta^{-1} tr(\nabla u \nabla)$
- M_t is the Martingale

$$
M_t = exp\{-\int_0^t u(B_s, s) \cdot dB_s - \frac{1}{2} \int_0^t |u(B_s, s)|^2 ds\}
$$

- Using M_t as an integrating factor eliminates the inertial terms from the equation (1)

Cameron-Martin and Feynmann-Kac

- The Feynmann-Kac formula gives the exponential of a sum of terms of the form

$$\int_s^t dq^k = \int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz),$$

  by a computation similar to the one that produces the geometric Lévy process, m^k the Lévy measure.

- The form of the processes

$$e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)} = e^{N_t^k \ln \beta + \gamma \ln|k|} = |k|^\gamma \beta^{N_t^k}$$

  was found by She and Leveque [7], for $h_k = \beta - 1$

- It was pointed out by She and Waymire [8] and by Dubrulle [4] that they are log-Poisson processes.

Outline

Computation of the structure functions

Lemma (The Kolmogorov-Obukhov-She-Leveque scaling)

The scaling of the structure functions is

$$S_p \sim C_p|x-y|^{\zeta_p},$$

where

$$\zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3})$$

$\frac{p}{3}$ *being the Kolmogorov scaling and* τ_p *the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's* $4/5$ *law,*

$$S_3 = -\frac{4}{5}\varepsilon|x-y|$$

to leading order, were $\varepsilon = \frac{d\mathcal{E}}{dt}$ *is the energy dissipation*

The first few structure functions

$$S_1(x,y,\infty) \leq \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|d_k|(1-e^{-\lambda_k t})}{|k|^{\zeta_1}} |\sin(\pi k \cdot (x-y))|$$

We get a stationary state as $t \to \infty$, and for $|x-y|$ small,

$$S_1(x,y,\infty) \sim \frac{2\pi^{\zeta_1}}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |d_k||x-y|^{\zeta_1},$$

where $\zeta_1 = 1/3 + \tau_1 \approx 0.37$. Similarly,

$$S_2(x,y,\infty) \sim \frac{4\pi^{\zeta_2}}{C^2} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} [{d_k}^2 + (\frac{C}{2})c_k]|x-y|^{\zeta_2},$$

when $|x-y|$ is small, where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$.

Higher order structure functions

Similarly,

$$S_3(x,y,\infty) \sim \frac{2^3\pi}{C^3} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} [|d_k|^3 + 3(C/2)c_k|d_k|]|x-y|.$$

For the pth structure functions, we get that S_p is estimated by

$$S_p \leq \frac{2^p}{C^p} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\sigma^p \cdot (-i\sqrt{2}\text{sgn}\, M)^p\, U\left(-\frac{1}{2}p, \frac{1}{2}, -\frac{1}{2}(M/\sigma)^2\right)}{|k|^{\zeta_p}} \times |\sin^p(\pi k\cdot(x-y))|.$$

where U is the confluent hypergeometric function, $M = |d_k|(1-e^{-\lambda_k t})$ and $\sigma = \sqrt{(C/2)c_k(1-e^{-2\lambda_k t})}$.

KOSL Scaling of the Structure Functions, higher order $Re_\lambda \sim 16,000$

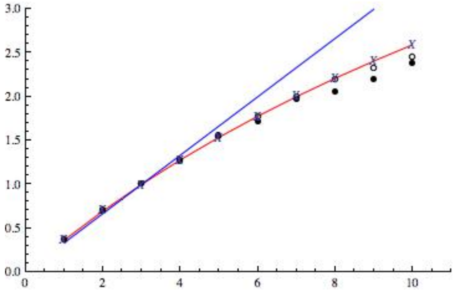

Figure: The exponents of the structure functions as a function of order, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [3], and experiments (X), from [7]. The Kolomogorov-Obukhov '41 scaling is also shown as a blue line for comparion.

KOSL Scaling of the Structure Functions, low order $Re_\lambda \sim 16,000$

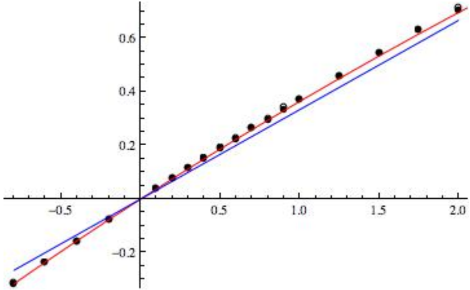

Figure: The exponents of the structure functions as a function of order $(-1, 2]$, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [3]. The Kolmogorov-Obukov '41 scaling is also shown as a blue line for comparion.

Outline

The Invariant Measure and the Probability Density Functions (PDF)

- Hopf [5] write down a functional differential equation for the characteristic function of the invariant measure
- The Kolmogorov-Hopf equation for (1) is

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{tr}[P_t C P_t^* \Delta \phi] + \mathrm{tr}[P_t \bar{D} \nabla \phi] + < K(z) P_t, \nabla \phi > \quad (1)$$

where $\bar{D} = (|k|^{1/3} D_k)$, $\phi(z)$ is a bounded function of z,

$$P_t = e^{-\int_0^t \nabla u \; dr} M_t \prod_k^m |k|^{2/3} (2/3)^{N_t^k}$$

- Variance and drift

$$Q_t = \int_0^t e^{K(s)} P_s C P_s^* e^{K^*(s)} ds, \quad E_t = \int_0^t e^{K(s)} P_s \bar{D} ds \quad (2)$$

The invariant measure of the stochastic Navier-Stokes

The solution of the Kolmogorov-Hopf equation (1) is

$$R_t\phi(z) = \int_H \phi(e^{Kt}P_t z + EI + y)\mathcal{N}_{(0,Q_t)} * \mathbb{P}_{N_t}(dy)$$

Theorem

The invariant measure of the Navier-Stokes equation on $H_c = H^{3/2^+}(\mathbb{T}^3)$ is, $\mu(dx) =$

$$e^{<Q^{-1/2}EI,\ Q^{-1/2}x>-\frac{1}{2}|Q^{-1/2}EI|^2}\mathcal{N}_{(0,Q)}(dx)\sum_k \delta_{k,l}\sum_{j=0}^{\infty} p^j_{m_l}\delta_{(N_l-j)}$$

where $Q = Q_\infty$, $E = E_\infty$.

The differential equation for the PDF

- The quantity that can be compared directly to experiments is the PDF

$$E(\delta_j u) = E([u(x+s,\cdot) - u(x,\cdot)] \cdot r) = \int_{\infty}^{\infty} x f_j(x) dx,$$

  $j = 1$, if $r = \hat{s}$ is the longitudinal direction, and $j = 2$, $r = \hat{t}$, $t \perp s$ is a transversal direction
- We take the trace of the Kolmogorov-Hopf equation (1)
- The stationary equation satisfied by the PDF is

$$\frac{1}{2}\phi_{rr} + \frac{1+|c|}{r}\phi_r = \frac{1}{2}\phi. \qquad (3)$$

Outline

The Probability Density Function (PDF)

Lemma

The PDF is a Normalized Inverse Gaussian distribution NIG of Barndorff-Nilsen [1]:

$$f(x_j) = \frac{(\delta/\gamma)}{\sqrt{2\pi}K_1(\delta\gamma)} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x_j-\mu)^2}\right) e^{\beta(x-\mu)}}{\left(\sqrt{\delta^2 + (x_j-\mu)^2}/\alpha\right)} \tag{4}$$

where K_1 is modified Bessel's function of the second kind, $\gamma = \sqrt{\alpha^2 - \beta^2}$.

$$f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{\Gamma(1) 2 e^{\beta\mu}}{(\delta^2 + (x-\mu)^2)}, \quad \text{for } x << 1$$

$$f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{e^{\beta(x-\mu)} e^{-\alpha x}}{x^{3/2}}, \quad \text{for } x >> 1$$

Conclusions

- The stochastic Navier-Stokes equations for the small scales in turbulence *with generic noise* can be used to construct the statistical theory of turbulence
- The additive noise is constructed by the Central Limit Theorem and the Large Deviation Principle. It consists of a general homogeneous Lévy process
- Multiplicative noise, consisting of simple jumps multiplying u gives, by the Feynmann-Kac formula, the log-Poisson processes of She-Leveque, Waymire and Dubrulle
- The estimate of the structure functions gives the Kolmogorov-Obukhov-She-Leveque scaling

More Conclusions

- The Kolmogorov-Hopf equation of the stochastic Navier-Stokes equation is found and its solution gives the invariant measure of turbulence
- The measure gives the NIG distributions as PDFs when projected, however, different moments have different parameters
- The non-uniqueness of solutions of the Navier-Stokes equation in the three dimensional case *is not a problem*
- The invariant measure exists by Leray's '34 [6] theory. If the velocity is not unique different velocities give equivalent statistics
- All of us together should be able to use this theory to improve numerical methods to the desired accuracy

The Artist by the Water's Edge
Leonardo da Vinci Observing Turbulence

Comparison with Simulations and Experiments

- We now compare the above PDFs with the PDFs found in simulations and experiments.
- The direct Navier-Stokes (DNS) simulations were provided by Michael Wilczek from his Ph.D. thesis, see [9].
- The experimental results are from Eberhard Bodenschatz experimental group in Göttingen.
- We thank both for the permission to use these results to compare with the theoretically computed PDFs.
- A special case of the hyperbolic distribution, the NIG distribution, was used by Barndorff-Nilsen, Blaesild and Schmiegel [2] to obtain fits to the PDFs for three different experimental data sets.

The PDF from simulations and fits for the longitudinal direction

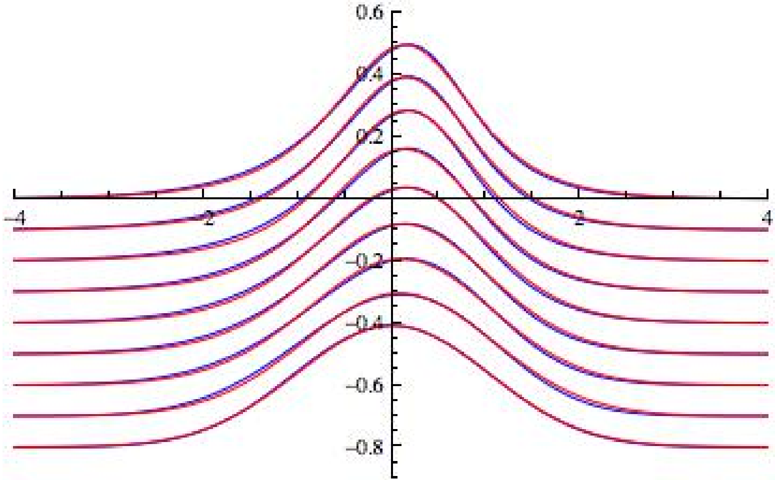

Figure: The PDF from simulations and fits for the longitudinal direction.

The log of the PDF from simulations and fits for the longitudinal direction

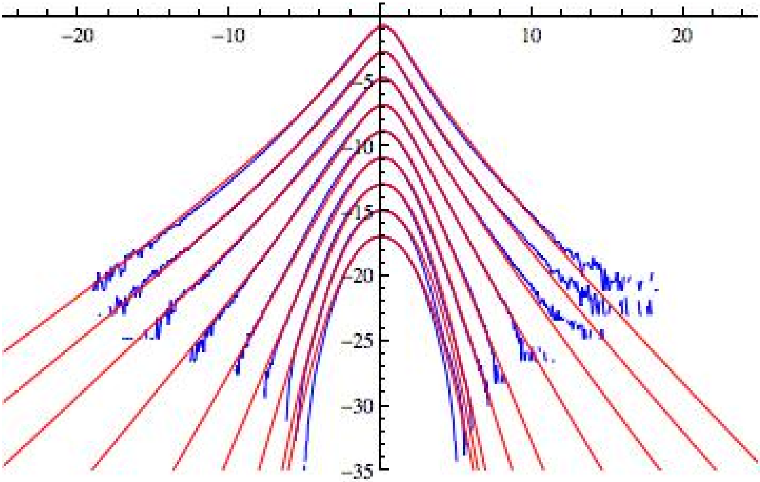

Figure: The log of the PDF from simulations and fits for the longitudinal direction, compare Fig. 4.5 in [9].

The PDF from simulations and fits for the transversal direction

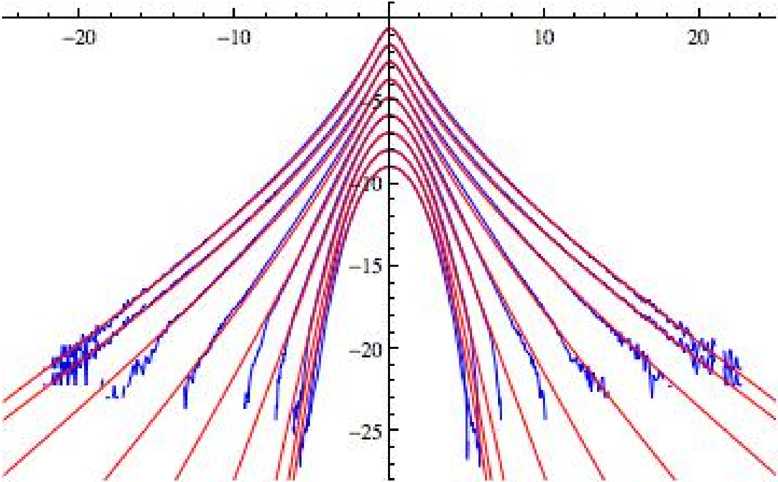

Figure: The log of the PDF from simulations and fits for the a

The PDFs from experiments and fits

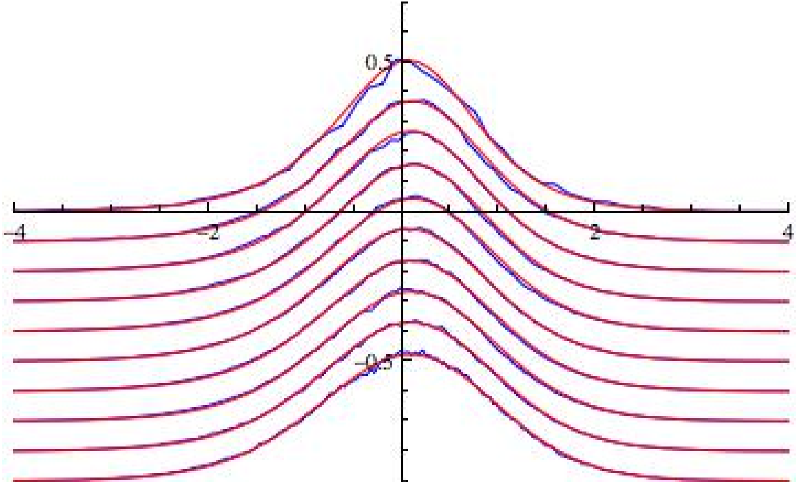

Figure: The PDFs from experiments and fits.

The log of the PDFs from experiments and fits

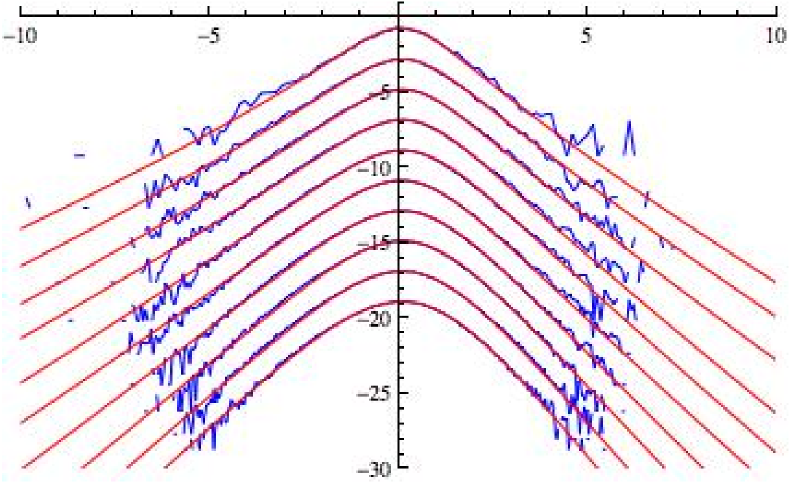

Figure: The log of the PDFs from experiments and fits.

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